Journal of Statistical Physics, Vol. 46, Nos. 3/4, 1987

Chapman–Enskog as an Application of the Method for Eliminating Fast Variables

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Received July 23, 1986; revision received November 10, 1986

Although the Chapman–Enskog treatment of the Boltzmann equation is one of the first examples of the elimination of fast variables, it is not usually presented consistently from that point of view. Here it is developed systematically as a special case of the general method for eliminating fast variables from nonlinear equations. As a result certain ambiguities can be remedied. First, it is inconsistent with the separation of time scales to extend the phenomenological description of the gas by including some of the higher moments of the distribution function (such as the heat flow). In the application to the relativistic Boltzmann equation, the dilemma concerning the choice of the lowest order approximation is resolved. In the final section it is demonstrated that unsystematic elimination of fast variables leads to secular terms.

KEY WORDS: Kinetic theory; elimination of fast variables; extended thermodynamics; relativistic Chapman–Enskog method.

1. INTRODUCTION AND SUMMARY

The celebrated Chapman–Enskog treatment of the Boltzmann equation (1–6) is not just a method for obtaining an approximate solution. It also manages to extract from the kinetic equation for the particle distribution $F(t, \mathbf{r}, \mathbf{p})$ a set of hydrodynamic equations for the particle number $n(\mathbf{r})$, the momentum $\mathbf{g}(\mathbf{r})$, and the energy $e(\mathbf{r})$ per unit volume. These quantities constitute a drastically reduced specification of the state of the gas: the other variables needed to specify $F(t, \mathbf{r}, \mathbf{p})$ have been eliminated. This is possible because all other variables vary rapidly—on the time scale of the collisions—while n, $\mathbf{g}$, and e are not affected by the collisions. They vary only through transport, which is slow when the gradients are small. The

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0022-4715/87/0200-0709$05.00/0

time scales are separated by a parameter ε, roughly the ratio of the mean free path to the distance over which $F(t, \mathbf{r}, \mathbf{p})$ varies appreciably in space.

The general problem of eliminating fast variables has received much attention in recent years. Most of the work is concerned with linear equations[7,8] and is therefore not suitable for the Boltzmann equation. However, nonlinear equations have also been studied[9,10] and a general scheme has been developed.[10] My aim is to show that the Chapman–Enskog treatment of gases can be regarded as a straightforward application of that general scheme. Of course, the fact that Chapman–Enskog is a case of eliminating fast variables is well known. (The name has even been used generically for the general case.[11]) Yet a detailed derivation will lead to several conclusions.

The first conclusion is that the Chapman–Enskog method is not—in the words used by Grad[3] to describe the presentation by Chapman and Cowling[2]—"an *ad hoc* recipe for juggling terms in an expansion of F in order to get results which have the desired form." Rather, it is uniquely determined by the rules of the general scheme. In particular, the choice of the slow variables n, $\mathbf{g}$, and e is decreed by the form of the Boltzmann equation itself rather than by the desire to recover the familiar hydrodynamic equations. This implies the second conclusion: There is no foundation for including additional variables in the macroscopic description, as is done in so-called "extended thermodynamics."[12]

Third, the applications of the Chapman–Enskog method to relativistic gases[13–16] suffer from an irritating arbitrariness in the definition of the hydrodynamic quantities and hence of the resulting hydrodynamic equations. This lack of uniqueness indicates that an essential ingredient is missing. I claim that what is missing is the proper separation of time scales. It will be shown in Section 4 that the systematic scheme for eliminating fast variables leads to a perfectly unique definition. It does not coincide with any of the options treated in the literature. In fact, it is not Lorentz-invariant: Two different observers looking at the same gas will extract different sets of slow variables, not related to each other by a Lorentz transformation!

2. THE GENERAL METHOD FOR ELIMINATING FAST VARIABLES

I summarize the relevant part of Ref. 10. Consider a set of J differential equations for J unknown x_j, involving a small parameter ε:

$$dx_j/d\tau = f_j(x; \varepsilon), \qquad j = 1, 2, \ldots, J \tag{2.1}$$

It is supposed that the f_j can be at least formally expanded in ε. A slow

variable is a function $Y(x)$ such that $dY/dt = O(\varepsilon)$. The general method of eliminating fast variables aims at deriving equations for the evolution of the slow variables alone, in which the other variables no longer appear. That is achieved by the following four steps.

(i) Identify the slow variables by solving the "unperturbed" equation

$$dx_j/d\tau = f_j(x; 0) \tag{2.2}$$

and determining its constants of the motion, i.e., the functions $Y(x)$ such that

$$\frac{dY}{d\tau} \equiv \sum_j \frac{\partial Y}{\partial x_j} f_j(x; 0) = 0 \tag{2.3}$$

They are the slow variables of (2.1). Let there be a complete set of R independent $Y_r(x)$ [i.e., all other functions obeying (2.3) can be expressed as functions of these Y_r, but not of a smaller set].

(ii) Define as new variables, instead of the x_j,

$$y_r = Y_r(x), \qquad z_v = Z_v(x), \qquad v = 1, 2, \ldots, N \tag{2.4}$$

where the Z_v are arbitrary, subject only to the condition that the transformation from $\{x_j\}$ to $\{y_r, z_v\}$ can be inverted. Hence, $N = J - R$. Transformation of (2.1) yields

$$dy_r/d\tau = \varepsilon g_r(y, z; \varepsilon), \qquad dz_v/d\tau = h_v(y, z; \varepsilon) \tag{2.5}$$

where again g_r and h_v can be expanded in ε.

(iii) Rewrite these equations on the slow time scale $t = \varepsilon\tau$, on which the y_r vary,

$$dy_r/dt = g_r(y, z) \tag{2.6a}$$

$$dz_v/dt = (1/\varepsilon)\, h_v(y, z) \tag{2.6b}$$

We have here omitted for simplicity the ε dependence of g_r and h_v because it will not occur in the present application to the Chapman–Enskog problem. Now expand the z_v,

$$z_v = z_v^{(0)} + \varepsilon z_v^{(1)} + \varepsilon^2 z_v^{(2)} + \cdots$$

leaving the y_r as they are. Determine $z_v^{(0)}$ from

$$h_v(y, z^{(0)}) = 0 \tag{2.7}$$

It is assumed that these N equations have a single solution

$$z_\nu^{(0)} = \varphi_\nu(y) \tag{2.8}$$

and that this is a point attractor of (2.6b) when y is kept fixed. (This assumption characterizes what was called in Ref. 10 "the first category.") The slow motion of y_ρ is now given in lowest approximation by

$$dy_\rho/dt = g_\rho(y, \varphi(y)) \tag{2.9}$$

(iv) For the next approximation determine $z_\nu^{(1)}$ from the next order of (2.6b),

$$\frac{dz_\nu^{(0)}}{dt} = \frac{\partial h_\nu(y, z^{(0)})}{\partial z_\mu^{(0)}} z_\mu^{(1)} \qquad \text{(summation implied)} \tag{2.10}$$

Substitute the result again in (2.6a),

$$dy_\rho/dt = g_\rho(y, z^{(0)} + \varepsilon z^{(1)}) \tag{2.11}$$

This gives the slow motion one order better than (2.9).

It is clear that one can continue in this way to higher orders. It should be emphasized, however, that *the slow variables y_ρ have been picked once for all*: they are the constants of the motion of the unperturbed equation (2.2). The elimination of the fast variables leads to a closed set of equations (2.9); the higher orders add corrections to these equations as in (2.11), but do not have the effect of enlarging the set of slow variables. Thus, the elimination of fast variables yields (in each order) a closed set of equations for the evolution of the slow variables by themselves. The fast variables ride along as determined by (2.7) and (2.10).

The solutions found in this way constitute a *subclass* of all solutions of the original equations given in (2.1) or in (2.6), namely the subclass of those solutions that can be expanded in ε. However, in order to be able to solve (2.1) for every set of initial values $x_j(0)$ one needs *all* solutions.[17] The solutions of the subclass are specified by the initial values $y_\rho(0)$ alone, while the initial $z_\nu(0)$ cannot be chosen, but must obey (2.7) and (2.10). The reason why nonetheless this subclass has a dominant role is that every solution of (2.1) tends rapidly toward one of the solutions of the subclass. For, since $\varphi_\nu(y)$ is by assumption an attractor of (2.6b), every z_ν that initially violates (2.7) approaches the required value (2.8) in a short initial transient period of order $t \sim \varepsilon$. This transient (or "initial slip" or "boundary layer") is not included in the results (2.9), or (2.11), or any higher approximations. As a consequence, when applying this scheme to the Chapman–Enskog treatment of the Boltzmann equation we shall find the

hydrodynamic equations. But if some arbitrary initial $F(0, \mathbf{r}, \mathbf{p})$ is given, the initial nonhydrodynamic behavior eludes us during a transient period of a few collision times.

3. APPLICATIONS TO THE CHAPMAN–ENSKOG TREATMENT OF THE BOLTZMANN GAS

Let $\mathbf{r}$ be the position of a particle, $\mathbf{p}$ its momentum, and $\mathbf{v} = \mathbf{p}/m$ its velocity. Let $F(t, \mathbf{r}, \mathbf{p})\, d\mathbf{r}\, d\mathbf{p}$ be the number of particles at time t in a six-dimensional cell $d\mathbf{r}\, d\mathbf{p}$ of the one-particle phase space. The Boltzmann equation is

$$\frac{\partial F(t, \mathbf{r}, \mathbf{p})}{\partial t} = -\mathbf{v} \cdot \frac{\partial F(t, \mathbf{r}, \mathbf{p})}{\partial t} + \frac{1}{\varepsilon} C[F] \tag{3.1}$$

$C[F]$ is a nonlinear operator that maps any function $F(\mathbf{r}, \mathbf{p})$ into another function of $\mathbf{r}, \mathbf{p}$:

$$C[F] = \int w(\mathbf{p}, \mathbf{p}_1 | \mathbf{p}_2, \mathbf{p}_3)[F(\mathbf{r}, \mathbf{p}_2)\, F(\mathbf{r}, \mathbf{p}_3) - F(\mathbf{r}, \mathbf{p})\, F(\mathbf{r}, \mathbf{p}_1)]\, d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{p}_3 \tag{3.2}$$

The transition probability w has the form

$$w(\mathbf{p}, \mathbf{p}_1 | \mathbf{p}_2, \mathbf{p}_3) = \delta^3(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)\, \delta\left(\frac{\mathbf{p}^2}{2m} + \frac{\mathbf{p}_1^2}{2m} - \frac{\mathbf{p}_2^2}{2m} - \frac{\mathbf{p}_3^2}{2m}\right) \sigma \tag{3.3}$$

where σ is a function of $|\mathbf{p} - \mathbf{p}_1|$ and $(\mathbf{p} - \mathbf{p}_1) \cdot (\mathbf{p}_2 - \mathbf{p}_3)$, which describes the dynamics of two-particle collisions.[4] The function w has the symmetry properties

$$w(\mathbf{p}, \mathbf{p}_1 | \mathbf{p}_2, \mathbf{p}_3) = w(\mathbf{p}_1, \mathbf{p} | \mathbf{p}_2, \mathbf{p}_3) = w(\mathbf{p}, \mathbf{p}_1 | \mathbf{p}_3, \mathbf{p}_2) = w(\mathbf{p}_2, \mathbf{p}_3 | \mathbf{p}, \mathbf{p}_1)$$

The factor $1/\varepsilon$ is a scale factor for the overall density; small ε gives high density and therefore frequent collisions.

An alternative way of introducing ε is the following. Start from the Boltzmann equation without ε:

$$\partial F/\partial \tau = -\mathbf{v} \cdot \partial F/\partial \mathbf{r} + C[F]$$

One tries to solve it for the case that F varies slowly in space. This is expressed by setting

$$F(\mathbf{r}, \mathbf{p}) = F^*(\varepsilon \mathbf{r}, \mathbf{p}) = F^*(\mathbf{r}^*, \mathbf{p})$$

so that

$$\partial F^*/\partial\tau = -\varepsilon \mathbf{v}\cdot\partial F^*/\partial \mathbf{r}^* + C[F]$$

This is the same as (3.1) on the slow time scale $t=\varepsilon\tau$ if one drops the asterisks.

Both methods of introducing ε are equivalent because the Boltzmann equation is invariant for simultaneous rescaling of F and $(t, \mathbf{r})$. Note that ε is a physical quantity: the ratio of mean free path and variation length. *It is not just a bookkeeping parameter*, but can be varied by increasing the density or reducing the gradients. Hence, it makes sense to consider the limit $\varepsilon\to 0$; the separation of time scales is well-defined and physically meaningful. This limit is not incompatible with the Boltzmann equation itself, because the validity of this equation is based on the smallness of the density multiplied by the volume of a molecule.

Equation (3.1) is of the type (2.1) if one identifies $\mathbf{r}, \mathbf{p}$ with the subscript j, and $F(\mathbf{r}, \mathbf{p})$ with the unknown x_j. We want to eliminate the fast motion of the molecules so as to be left with the slow motion. Our first task is to find the constants of the motion of the unperturbed equation (2.2), which in the present case is

$$\partial F(\mathbf{r}, \mathbf{p})/\partial\tau = C[F] \tag{3.4}$$

According to (2.3), they must be functionals $Y[F]$ having the property

$$\int d\mathbf{r}\, d\mathbf{p}\, \frac{\delta Y[F]}{\delta F(\mathbf{r}, \mathbf{p})}\, C[F] = 0 \tag{3.5}$$

where δ denotes the functional derivative. We know that there are five "collisional invariants"

$$\psi_0 = \mathbf{p}^2/2m, \qquad \psi_{1,2,3} = \mathbf{p}, \qquad \psi_4 = 1$$

We also know that for a simple gas of particles without internal degrees of freedom these are the only ones.[18] Hence, we have at each space point $\mathbf{r}$ five quantities (3.5), namely the five moments

$$\begin{aligned} e(\mathbf{r}) &= \int (\mathbf{p}^2/2m)\, F(\mathbf{r}, \mathbf{p})\, d\mathbf{p} \\ \mathbf{g}(\mathbf{r}) &= \int \mathbf{p} F(\mathbf{r}, \mathbf{p})\, d\mathbf{p} \\ n(\mathbf{r}) &= \int F(\mathbf{r}, \mathbf{p})\, d\mathbf{p} \end{aligned} \tag{3.6}$$

This fivefold infinity constitutes the complete set of independent slow variables y_r.

For the fast variables z_v one may take the remaining moments of F at each space point. They vary on the time scale of the collisions, whereas the y_r vary only through transport. Of course one may choose for the z_v any other set of parameters that identify $F(\mathbf{r}, \mathbf{p})$ after the values of its moments (3.6) are fixed. Actually, we shall not need to make any explicit choice for the z_v, but we shall use the term "higher moments" for concreteness.

Next we have to solve (2.7), or, in the present case,

$$C[F^{(0)}] = 0 \tag{3.7}$$

From this equation the higher moments $z_v^{(0)}$ have to be found *for prescribed values of the five slow moments* (3.6). The solution is the local Maxwellian

$$F^{(0)}(\mathbf{r}, \mathbf{p}) = \exp[-\alpha(\mathbf{r}) - \gamma(\mathbf{r}) \cdot \mathbf{p} - \beta(\mathbf{r})\, \mathbf{p}^2/2m] \tag{3.8}$$

where $\alpha(\mathbf{r})$, $\gamma(\mathbf{r})$, and $\beta(\mathbf{r})$ are adjusted to give the prescribed values of $e(\mathbf{r})$, $\mathbf{g}(\mathbf{r})$, and $n(\mathbf{r})$:

$$e^{-\alpha} \int (\mathbf{p}^2/2m) \exp(-\gamma \cdot \mathbf{p} - \beta p^2/2m)\, d\mathbf{p} = e(\mathbf{r}) \tag{3.9a}$$

$$e^{-\alpha} \int \mathbf{p} \exp(-\gamma \cdot \mathbf{p} - \beta p^2/2m)\, d\mathbf{p} = \mathbf{g}(\mathbf{r}) \tag{3.9b}$$

$$e^{-\alpha} \int \exp(-\gamma \cdot \mathbf{p} - \beta p^2/2m)\, d\mathbf{p} = n(\mathbf{r}) \tag{3.9c}$$

The H-theorem provides the Liapunov function that guarantees that (3.8) is the unique attractor for fixed values of the slow variables e, $\mathbf{g}$, and n.

The analog of the equations (2.6a) are the conservation equations for the moments n, $\mathbf{g}$, and e obtained from (3.1) by integrating

$$\frac{\partial n(t, \mathbf{r})}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot \int \frac{\mathbf{p}}{m} F(t, \mathbf{r}, \mathbf{p})\, d\mathbf{p} \tag{3.10a}$$

$$\frac{\partial \mathbf{g}(t, \mathbf{r})}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot \int \frac{\mathbf{p}}{m} \mathbf{p} F(\mathbf{t}, \mathbf{r}, \mathbf{p})\, d\mathbf{p} \tag{3.10b}$$

$$\frac{\partial e(t, \mathbf{r})}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot \int \frac{\mathbf{p}}{m} \frac{p^2}{2m} F(t, \mathbf{r}, \mathbf{p})\, d\mathbf{p} \tag{3.10c}$$

The analog of (2.9) is obtained by substituting for F its lowest approximation (3.8). The result is the Euler equations for an ideal gas, as is well known.

Note that this result appears as the *lowest order equation* (3.8) in the process of eliminating fast variables. In the usual Chapman–Enskog treatment it appears as the integrability condition for the next-order equation. The explanation is that this integrability condition expresses the requirement that one is dealing with a solution that can be expanded in ε, and this requirement is already implied in the general scheme of Section 2.

To obtain the next approximation, the corrections $z_\nu^{(1)}$ of the fast moments have to be computed from (2.10). A slight complication is that we do not know the $z_\nu^{(0)}$ explicitly; rather, they are implicitly given as the moments of (3.8). It is therefore convenient to combine (2.10) with the equation (2.9) for the slow moments (which has previously been satisfied), so as to obtain an equation for *all* moments, that is, for the entire first-order correction $F^{(1)}$ of the distribution. That amounts to extracting the terms of order ε^0 from (3.1):

$$\frac{\partial F^{(0)}(\mathbf{r}, \mathbf{p})}{\partial t} = -\mathbf{v} \cdot \frac{\partial F^{(0)}(\mathbf{r}, \mathbf{p})}{\partial \mathbf{r}} + \int d\mathbf{r}'\, d\mathbf{p}' \frac{\delta C[F^{(0)}]}{\delta F^{(0)}(\mathbf{r}', \mathbf{p}')} F^{(1)}(\mathbf{r}', \mathbf{p}') \quad (3.11)$$

From this equation the fast moments $z_\nu^{(1)}$ in $F^{(1)}$ have to be determined without altering the values of the slow moments $y_r = \{n(\mathbf{r}), \mathbf{g}(\mathbf{r}), e(\mathbf{r})\}$. Hence one must solve (3.11) with the supplementary conditions

$$\int (\mathbf{p}^2/2m)\, F^{(1)}\, d\mathbf{p} = 0, \qquad \int \mathbf{p} F^{(1)}\, d\mathbf{p} = 0, \qquad \int F^{(1)}\, d\mathbf{p} = 0 \quad (3.12)$$

Equation (3.11) is a linear, inhomogeneous integral equation for $F^{(1)}$. The integrability condition is equivalent to the Euler equations, which have been satisfied in the previous approximation. Hence the solution exists and it is made unique by the additional requirements (3.12). The corrected equation (2.11) for the slow variables is obtained by inserting the corrected $F = F^{(0)} + \varepsilon F^{(1)}$ found in this way into (3.10). The result is the Navier–Stokes equations. The only difficulty is that no explicit solution of (3.11) is known (except in the case of Maxwell molecules). Nor does the equation involve a small parameter on which to base a systematic expansion, so that one has to invoke various less systematic approximation methods.[2,6] If one expresses (3.11) as an equation of the moments, one obtains an infinite hierarchy of equations for the infinite sequence of $z_\nu^{(1)}$.

All this is of course precisely the Chapman–Enskog scheme. We only wanted to demonstrate that the scheme *follows uniquely from the general method for eliminating fast variables.* It is also seen that the higher orders in ε merely add corrections to the five equations for n, $\mathbf{g}$, and e and *never lead to the addition of equations for additional slow (or perhaps semislow) quantities.* The systematic elimination of slow variables does not justify singling

out the heat flow or the off-diagonal elements of the pressure tensor[18,3] for special treatment. All z_ν vary rapidly on the time scale of the collisions. The expansion in ε leads uniquely to n, $\mathbf{g}$, and e as the only quantities whose rate of variation is determined by the gradients. It does not lead to "extended non-equilibrium thermodynamics."[12]

4. THE RELATIVISTIC CHAPMAN–ENSKOG TREATMENT

Greek labels μ, ν, λ take the values 0, 1, 2, 3, and Roman labels k, l take the values 1, 2, 3. The metric tensor is

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -1) = g^{\mu\nu}$$

We write $x^\mu = (t, \mathbf{r})$, $p^\mu = ((\mathbf{p}^2 + m^2)^{1/2}, \mathbf{p})$, and $\mathbf{p}/p^0 = \mathbf{v}$. Then $d\mathbf{p}/p^0$ is invariant and so is the particle density in phase space $F(t, \mathbf{r}, \mathbf{p}) \equiv F(x, \mathbf{p})$. More precisely, a particle at the space-time point x with momentum $\mathbf{p}$ is seen by another observer at x' with momentum $\mathbf{p}'$; his particle density F' is related to F by $F'(x', \mathbf{p}') = F(x, \mathbf{p})$. The four-vector

$$N^\mu(x) = \int p^\mu F(x, \mathbf{p})\, d\mathbf{p}/p^0 = (n, n\langle \mathbf{v} \rangle) \tag{4.1}$$

represents the particle density and flow in space. The tensor

$$T^{\mu\nu}(x) = \int p^\mu p^\nu F(x, \mathbf{p})\, d\mathbf{p}/p^0 \tag{4.2}$$

represents the energy density $T^{00} = e(x)$, the momentum density $T^{0k} = g^k(x)$, and the stresses T^{kl}.

The relativistic Boltzmann equation is the same as (3.1), but for two modifications. First, one now has $\mathbf{v} = \mathbf{p}/p^0$. Second, (3.2) is to be replaced with

$$w(\mathbf{p}, \mathbf{p}_1 | \mathbf{p}_2, \mathbf{p}_3) = \delta^4(p^\lambda + p_1^\lambda - p_2^\lambda - p_3^\lambda)\, \sigma \tag{4.3}$$

where σ is to be computed by means of the relativistic dynamics of two-body collisions.[14,16] Thus, the relativistic Boltzmann equation is

$$\frac{\partial F(x, \mathbf{p})}{\partial t} = -\frac{p^k}{p^0}\frac{\partial F}{\partial x^k} + \frac{1}{\varepsilon} C[F] \tag{4.4}$$

where $C[F]$ again has the form (3.2) with w as in (4.3). It is customary to write the equation in a manifestly covariant way (see Section 5), but that is neither necessary nor convenient for our purpose. The reason is that

eliminating fast variables is an inherently noncovariant approximation process since it singles out the time coordinate.

To eliminate the fast variables, we again first have to identify the slow variables defined by (3.5). The collisional invariants corresponding to (4.3) are

$$\psi_0 = (\mathbf{p}^2 + m^2)^{1/2}, \qquad \psi_{1,2,3} = \mathbf{p}, \qquad \psi_4 = 1$$

and they are again the only ones.[14] Hence, we have again at each point $\mathbf{r}$ five slow variables, namely the five moments

$$g^\mu(x) = \int p^\mu F(x, \mathbf{p})\, d\mathbf{p}, \qquad n(x) = \int F(x, \mathbf{p})\, d\mathbf{p} \tag{4.5}$$

They obey the equations [obtained by multiplying (4.4) with the several ψ and integrating]

$$\frac{\partial g^\mu}{\partial t} = -\frac{\partial}{\partial x^l} \int \frac{p^l}{p^0} p^\mu F(x, \mathbf{p})\, d\mathbf{p} \tag{4.6a}$$

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial x^l} \int \frac{p^l}{p^0} F(x, \mathbf{p})\, d\mathbf{p} \tag{4.6b}$$

These are the analogs of (2.6a).

The fast variables z_ν are the remaining moments of F. To obtain the zeroth approximation, one has to solve (2.7), that is, for given g^μ and n we have to determine the higher moments of F in such a way that (3.7) is obeyed. The solution is again the local equilibrium distribution, which is the Jüttner distribution[21,19]

$$F^{(0)}(x, \mathbf{p}) = \exp[-\alpha(x) - \beta_\lambda(x)\, p^\lambda] \tag{4.7}$$

The parameters α and β_λ have to be adjusted to obtain the prescribed g^μ and n (at each space-time point x):

$$e^{-\alpha} \int \exp(-\beta_\lambda p^\lambda)\, d\mathbf{p} = n = \int F\, d\mathbf{p} \tag{4.8a}$$

$$e^{-\alpha} \int p^k \exp(-\beta_\lambda p^\lambda)\, d\mathbf{p} = g^k = \int p^k F\, d\mathbf{p} \tag{4.8b}$$

$$e^{-\alpha} \int p^0 \exp(-\beta_\lambda p^\lambda)\, d\mathbf{p} = e = \int p^0 F\, d\mathbf{p} \tag{4.8c}$$

It is shown in the Appendix that there exists a unique solution α, β_λ.

The zeroth approximation (2.9) for the evolution of the slow variables is obtained by replacing F in (4.6) with $F^{(0)}$

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial x^l} e^{-\alpha} \int \frac{p^l}{p^0} \exp(-\beta_\lambda p^\lambda)\, d\mathbf{p} \tag{4.9a}$$

$$\frac{\partial g^k}{\partial t} = -\frac{\partial}{\partial x^l} e^{-\alpha} \int \frac{p^l p^k}{p^0} \exp(-\beta_\lambda p^\lambda)\, d\mathbf{p} \tag{4.9b}$$

$$\frac{\partial e}{\partial t} = -\frac{\partial}{\partial x^l} e^{-\alpha} \int p^l \exp(-\beta_\lambda p^\lambda)\, d\mathbf{p} \tag{4.9c}$$

Together with the equations (4.8), which connect α and β_λ with n, $\mathbf{g}$, and e, this is a closed set of equations, analogous to the nonrelativistic Euler equations.

In order to give this result a more familiar form, we set $(\beta_\lambda \beta^\lambda)^{1/2} = \beta$ and note that

$$T = 1/k_B \beta, \qquad U_\mu = \beta_\mu/\beta, \qquad \mu = -\alpha/\beta \tag{4.10}$$

are the temperature, four-velocity of the fluid, and chemical potential. For a gas in equilibrium these identifications follow uniquely from the general principles of statistical mechanics. For our nonequilibrium gas described by $F(x, \mathbf{p})$ the local values of $T(x)$, $U_\mu(x)$, and $\mu(x)$ are *defined* as those of the approximating local equilibrium distribution $F^{(0)}$. They are uniquely determined by (4.8).

We shall call (4.10) intensive variables and refer to n, $\mathbf{g}$, and e as "extensive" variables (although they are actually extensive quantities per unit volume). The equations of motion (4.9) express the rate of change of the extensive variables in terms of the gradients of the intensive ones. The equations of state, or "constituent equations" (4.8) are the link between the intensive and extensive variables needed to close the equations of motion. All these variables refer to the slow or phenomenological level of description. Note, however, that the equation of motion for e,

$$\frac{\partial e}{\partial t} = -\frac{\partial}{\partial x^l} \int p^l F\, d\mathbf{p} = -\frac{\partial g^l}{\partial x^l}$$

is already closed within the extensive variables, even without approximation.

The next approximation requires the solution of (2.10), which in the present case takes the form of the integral equation (3.11) for $F^{(1)}$ with the additional requirements

$$\int p^\lambda F^{(1)}\, d\mathbf{p} = 0, \qquad \int F^{(1)}\, d\mathbf{p} = 0$$

The integrability conditions are the lower order equations (4.9) previously satisfied. In the Chapman–Enskog theory this is usually called the "second" approximation. It results in the hydrodynamic equations, including expressions for the viscosity and heat conduction. It contrast with the non-relativistic case, there also results a *bulk viscosity*,[13] which is a consequence of the fact that $\mathbf{v}$ is not proportional to $\mathbf{p}$.

5. DISCUSSION OF THE RELATIVISTIC CASE

The literature on relativistic kinetic theory[13-16] utilizes a manifestly covariant form of the Boltzmann equation. One defines

$$W(\mathbf{p}, \mathbf{p}_1 | \mathbf{p}_2, \mathbf{p}_3) = p^0 p_1^0 p_2^0 p_3^0 w(\mathbf{p}, \mathbf{p}_1 | \mathbf{p}_2, \mathbf{p}_3)$$

which turns out to be invariant under a simultaneous Lorentz transformation of the p's. Then (4.4) may be written

$$p^\mu \frac{\partial F}{\partial x^\mu} = \frac{1}{\varepsilon} D[F] \tag{5.1a}$$

$$D[F] = \int W(\mathbf{p}, \mathbf{p}_1 | \mathbf{p}_2, \mathbf{p}_3)[F(\mathbf{p}_2) F(\mathbf{p}_3) - F(\mathbf{p}) F(\mathbf{p}_1)] \frac{d\mathbf{p}_1}{p_1^0} \frac{d\mathbf{p}_2}{p_2^0} \frac{d\mathbf{p}_3}{p_3^0} \tag{5.1b}$$

This equation is the same as (4.4), and therefore no more and no less covariant, but its covariance is manifest.

The existing accounts of the application of the Chapman–Enskog treatment to this relativistic Boltzmann equation are not based on a systematic elimination of the fast variables. Rather, one follows the non-relativistic paradigm and postulates, in analogy with (3.8), that the zeroth approximation is a local equilibrium distribution of the form (4.7). The question then arises of how to fit the parameters $\alpha(x)$ and $\beta_\lambda(x)$ to the actual $F(x, \mathbf{p})$. In the absence of a clear prescription, three options have emerged in the literature.

(i) Following Eckart,[22] one first defines as "rest frame" the one in which the mean velocity $\langle \mathbf{v} \rangle$, defined in (4.1), vanishes. In this rest frame one postulates for $F^{(0)}$ the expression (4.7) with $\beta_k = 0$ and subsequently requires β_0 and α to have the values that reproduce the actual energy and particle densities *in this frame*:

$$\int p^0 F(x, \mathbf{p})\, d\mathbf{p} = \int p^0 F^{(0)}(x, \mathbf{p})\, d\mathbf{p}$$

$$= e^{-\alpha} \int p^0 \exp(-\beta_0 p^0)\, d\mathbf{p} \tag{5.2a}$$

$$\int F(x, \mathbf{p})\, d\mathbf{p} = \int F^{(0)}(x, \mathbf{p})\, d\mathbf{p}$$

$$= e^{-\alpha} \int \exp(-\beta_0 p^0)\, d\mathbf{p} \tag{5.2b}$$

In all other frames the α has the same value and the β_λ is obtained by Lorentz transformation.

(ii) Following Landau and Lifshitz,[23] one chooses as rest frame the one in which $\langle \mathbf{p} \rangle$ vanishes, that is, $T^{0k} = 0$, and requires that in *this* frame the energy and particle densities match. (This option remains feasible if the particle number is not conserved.)

In both these options the fitting of $F^{(0)}$ to F is nonlinear in F. This does not appear from (5.2), but determining the rest frame and transforming to it is a highly nonlinear procedure. Option (i) is hard to generalize to mixtures, since there is no unique way to select a rest frame. The next option, although linear, has the same drawback.

(iii) Stewart[15] chose to fit $F^{(0)}$ to F so as first to reproduce the actual N^μ

$$\int p^\mu F^{(0)}(\mathbf{p}) \frac{d\mathbf{p}}{p^0} = \int p^\mu F(\mathbf{p}) \frac{d\mathbf{p}}{p^0}$$

This is the same requirement as used in (i). For the fifth condition needed to fix α and β_λ he requires that also the value of

$$T^\mu{}_\mu(x) = m^2 \int F(x, \mathbf{p})\, d\mathbf{p}/p^0 \tag{5.3}$$

is reproduced by $F^{(0)}$.

Each of these three options is constructed so as to be invariant, in the sense that two observers, looking at the same F, will fit the same $F^{(0)}$ to it. This is clear both for (i) and for (ii), because all observers are told to refer back to one or the other rest frame. Option (iii) achieves invariance in a more elegant way: it uses for the fitting the covariant N^μ and invariant trace (5.3).

The three options are not equivalent. They all start from the exact conservation laws (4.6) for the extensive variables; they all insert for F a

zeroth approximation $F^{(0)}$ of the form (4.7) to arrive at (4.9); but in each of them the intensive variables α and β_λ have a different meaning and are linked to the extensive variables by different equations of state. Thus, the three options assign to the same gas different values for the local velocity, density, and temperature. There are no reasons to prefer any one of them, only arguments of convenience or elegance. We have shown in Section 4 that the systematic elimination of fast variables leads inescapably to a fourth procedure—the word "option" being inappropriate.

This fourth procedure is not covariant: each observer approximates F by his own $F^{(0)}$. Formally, the reason is that the elimination of fast variables treats time as different from the space coordinates. Physically, it means that *what is observed as a slow variable by one observer does not appear as a slow variable to another.* The slow variables of one observer cannot be expressed in terms of those of the other, because each observer extracts different aspects of $F(x, \mathbf{p})$ as his private slow variables. It is true that $g^\mu = T^{\mu 0}$ transforms as the column of a tensor, but this transformation involves elements T^{kl}, which are fast variables and depend on the detailed function $F(x, \mathbf{p})$, not just on its slow aspects.

Incidentally, in general relativity one often assumes the existence of a relativistic fluid having a well-defined local four-velocity. That implies that locally the fluid is nonrelativistic. In each fluid element the particles have nonrelativistic velocities with respect to each other, but far away elements may have relativistically different velocities.

Finally, it may be remarked that adopting additional moments of F into the macroscopic description[24,15] is just as much a violation of the idea of eliminating fast variables as in the nonrelativistic case, Section 3.

6. THE NECESSITY OF STARTING WITH THE CORRECT SLOW VARIABLES

The lact of uniqueness in the usual treatments of the relativistic case is sometimes excused by the comment that differences in the choice of the zeroth order are automatically compensated by the higher order terms. It is true that, when the zeroth approximation is not based on the actual slow variables (4.5), one obtains compensating terms in the next order. However, this amounts to expanding not only the fast variables, but the slow variables as well. As a result, the compensating terms are secular, that is, they represent an expansion in (εt) rather than in ε. The aim of the method for eliminating fast variables is just to avoid such terms.

As an illustration of the occurrence of these terms, I here give another way of eliminating fast variables. It has the apparent advantage over the

one in Section 2 that no explicit transformation to slow and fast variables is required. That looks attractive because it avoids the "slight complication" mentioned in Section 3, namely that the fast variables $z_\nu^{(0)}$ are not given explicitly. However, the drawback is that the slow variables are not identified from the start and have to be corrected in the higher orders, which gives rise to the dreaded secular terms.

Consider Eq. (2.1) and its reduced or unperturbed form (2.2). In the latter equation the y_r are constants, and so are the $z_\nu^{(0)}$ that are found from (2.7). Suppose one transforms $(y_r, z_\nu^{(0)})$ back into the variables x_j and calls the result $x_j^{(0)}$, that is, according to (2.4),

$$y_r = Y_r(x^{(0)}), \qquad z_\nu^{(0)} = Z_\nu(x^{(0)})$$

Then one has

$$f_j(x^{(0)}; 0) = 0 \tag{6.1}$$

Thus, $x_\nu^{(0)}$ is an equilibrium point of (2.2) and since (2.7) has one solution for each set of values of the y_r, there is an R-dimensional manifold of such points. The aim is to find the slow motion along this manifold caused by the higher terms in (2.1).

Set

$$f_j(x; \varepsilon) = f_j^{(0)}(x) + \varepsilon f_j^{(1)}(x) + \cdots$$

and expand x_j in the same way. Write $t = \varepsilon\tau$ and collect successive orders of ε:

$$dx_j^{(0)}/dt = f_{j,k}^{(0)}(x^{(0)})\, x_k^{(1)} + f_j^{(1)}(x^{(0)}) \tag{6.2}$$

$$dx_j^{(1)}/dt = f_{j,k}^{(0)}(x^{(0)})\, x_k^{(2)} + \tfrac{1}{2} f_{j,kl}^{(0)}(x^{(0)})\, x_k^{(1)} x_l^{(1)} + f_{j,k}^{(1)}(x^{(0)})\, x_k^{(1)} \tag{6.3}$$

The subscripts k and l denote differentiations, and summation is implied. Equations (6.2) have to be solved for $x_j^{(0)}$ with the constraint (6.1); the $x_k^{(1)}$ serve as Lagrange parameters. The $J \times J$ matrix $f_{j,k}^{(0)}$ is of rank $N = J - R$, because it has R left null vectors according to (2.3). Thus, the J quantities $x_k^{(1)}$ correspond effectively to only N adjustable terms in (6.2), just enough to satisfy (6.1) as well. On the other hand, there is still an R-parameter freedom in the choice of $x_k^{(1)}$.

This freedom is the subject of (6.3). Here the $x_k^{(2)}$ are effectively N Lagrange parameters, just enough to solve (6.3) for $x_j^{(1)}$ subject to the N constraints that resulted from the solution of (6.2). In this way one finds $x_j(t)$ order by order. No explicit separation of slow and fast variables is needed.

However, the resulting equations (6.1)–(6.3) have a very different structure than those of Section 2. There the slow variables y_r obeyed to lowest order (2.9); the next order (2.11) provided an improvement of this equation. In the present approach the lowest order (6.2) with (6.1) describes a slow motion of $x_j^{(0)}$. The next order does not alter this equation, but adds a term to $x_j^{(0)}$ so as to obtain an improved $x_j = x_j^{(0)} + \varepsilon x_j^{(1)}$. This implies that the slow part of x_j is expanded in ε together with the fast part. As a consequence, the functions $f_j^{(0)}, f_j^{(1)}, \ldots$ are expanded around the uncorrected value $x_j^{(0)}$. For instance, the last term of (6.3) is correcting for the fact that $f_j^{(1)}$ in (6.2) was taken at the uncorrected $x_j^{(0)}$. This term is secular, because $x_j^{(1)}$ varies on the slow time scale.

When the method of this section is applied to the Boltzmann equation it appears to be just the Hilbert expansion (as noted by a referee), which is known to produce secular terms.[2] Our calculation serves only to *illustrate* how secular terms arise as a penalty for not starting out from the correct slow variables. To actually *prove* that the same is true for the Chapman–Enskog expansion that starts from one of the three options mentioned would require the explicit calculation of the second-order terms.

APPENDIX

We first prove that Eqs. (4.8) cannot have more than one solution α, β_λ. Divide by n in order to eliminate α,

$$\frac{g^k}{n} = \frac{\int p^k \exp(-\beta_\lambda p^\lambda)\, d\mathbf{p}}{\int \exp(-\beta_\lambda p^\lambda)\, d\mathbf{p}} = -\frac{\partial}{\partial \beta_k} \log Z \tag{A.1a}$$

$$\frac{e}{n} = \frac{\int p^0 \exp(-\beta_\lambda p^\lambda)\, d\mathbf{p}}{\int \exp(-\beta_\lambda p^\lambda)\, d\mathbf{p}} = -\frac{\partial}{\partial \beta_0} \log Z \tag{A.1b}$$

We have introduced the function

$$Z(\beta_\lambda) = \int \exp(-\beta_\lambda p^\lambda)\, d\mathbf{p} = \int \exp[-\beta_k p^k - \beta_0 (\mathbf{p}^2 + 1)^{1/2}]\, d\mathbf{p}$$

which exists for $\beta_0 > |\boldsymbol{\beta}|$ and is positive. The exponent is linear in β_λ and therefore (marginally) convex. Hence, the exponential itself is (by definition) logarithmically convex and the integral, being a sum of such functions, is also logarithmically convex. Thus, $\log Z$ is represented by a convex hypersurface over the space of the four variables β_λ. Equations (A.1) determine the point on the hypersurface in which the tangent hyperplane has the direction $e/n{:}g^k/n{:}1$. There cannot be more than one such point, because of the convexity. Hence, (A.1) cannot have more than one

solution β_λ. Subsequently, α has to be found from (4.8a) and is also uniquely determined.

Next we prove that no solutions of (4.8) exist unless

$$(e/n)^2 - (\mathbf{g}/n)^2 > 1 \tag{A.2}$$

For any two vectors $\mathbf{p}$ and $\mathbf{p}'$ one easily verifies

$$(\mathbf{p}^2+1)^{1/2}(\mathbf{p}'^2+1)^{1/2} - \mathbf{p}\cdot\mathbf{p}' \geqslant 1 \tag{A.3}$$

Let both vectors be random, independently distributed with the same probability ρ. Multiply (A.3) with $\rho(\mathbf{p})\,\rho(\mathbf{p}')$ and integrate:

$$\langle(\mathbf{p}^2+1)^{1/2}\rangle^2 - \langle\mathbf{p}\rangle^2 \geqslant 1 \tag{A.4}$$

The equality sign in (A.3) applies only if $\mathbf{p}'$ is equal to $\mathbf{p}$. Hence, (A.4) is a strict inequality unless $\rho(\mathbf{p})$ is a delta function. Take $\rho(\mathbf{p}) = [\exp(-\beta_\lambda p^\lambda)]/Z$ and (A.4) becomes the strict inequality (A.2). An alternative proof can be given using the convexity of the function $(p^2+1)^{1/2}$.[25]

Finally, we prove the existence of a solution α, β_λ subject to the conditions (A.2) and $n>0$. We may assume $\boldsymbol{\beta}$ parallel to $\mathbf{g}$ and take that direction as the z axis. Then there are two equations for two unknowns β_z and β_0,

$$\frac{g^z}{n} = -\frac{\partial}{\partial\beta_z}\log Z, \qquad \frac{e}{n} = -\frac{\partial}{\partial\beta_0}\log Z$$

$$Z(\beta_0,\beta_z) = \int \exp[-\beta_z p^z - \beta_0(p^2+1)^{1/2}]\,d\mathbf{p}$$

$$= \frac{2\pi}{\beta_z}\int_{-\infty}^{\infty} \{\exp[\beta_z p - \beta_0(p^2+1)^{1/2}]\}\; p\,dp \tag{A.5}$$

Z is defined inside the sector Δ given by $-\infty < \beta_z < \infty$, $\beta_0 > |\beta_z|$. We need to know the image of Δ in the (u, v) plane, where

$$u = -\frac{\partial \log Z}{\partial\beta_z}, \qquad v = -\frac{\partial \log Z}{\partial\beta_0}$$

Due to the convexity, the mapping is one-to-one and it suffices to find the images of the boundaries of Δ.

When both β_0 and β_z are large, the integral (A.5) is dominated by the maximum of the exponent at $p_m = \beta_z(\beta_0^2 - \beta_z^2)^{-1/2}$. One obtains asymptotically

$$Z = \frac{1}{(2\pi)^{1/2}} \frac{\beta_0}{(\beta_0^2 - \beta_z^2)^{5/4}} \exp[-(\beta_0^2 - \beta_z^2)^{1/2}]$$

$$u = -\frac{\beta_z}{(\beta_0^2 - \beta_z^2)^{1/2}} + O\left(\frac{1}{\beta_0}\right)$$

$$v = \frac{\beta_0}{(\beta_0^2 - \beta_z^2)^{1/2}} + O\left(\frac{1}{\beta_0}\right)$$

Thus, the hyperbola $v^2 - u^2 = 1$ is part of the boundary of the image of Δ. This coincides with the condition (A.2).

When β_0 and β_z are finite, the integral (A.5) diverges as β_0 approaches $|\beta_z|$, that is, on the boundary of Δ. Write

$$Z = \frac{2\pi}{\beta_z}\left[\int_{-A}^{A} + \int_{A}^{\infty} + \int_{-\infty}^{A}\right]$$

For fixed $A \gg 1$ the first integral is always finite and

$$\int_A^\infty \{\exp[\beta_z p - \beta_0(p^2+1)^{1/2}]\}\, p\, dp \simeq \int_A^\infty \{\exp[(\beta_z - \beta_0)\, p]\}\, p\, dp$$

$$= \frac{1}{(\beta_0 - \beta_z)^2}$$

where again finite terms are omitted. Hence

$$Z = \frac{2\pi}{\beta_z}\left[\frac{1}{(\beta_0 - \beta_z)^2} - \frac{1}{(\beta_0 + \beta_z)^2} + \text{finite terms}\right]$$

$$u = -\frac{4\beta_z}{\beta_0^2 - \beta_z^2}, \qquad v = \frac{4\beta_0}{\beta_0^2 - \beta_z^2}$$

Thus, the point u, v moves to infinity when the point β_0, β_z approaches the boundary of Δ.

The conclusion is that the interior of Δ is mapped onto the region in the u, v plane bounded by the hyperbola, that is, $v > (u^2+1)^{1/2}$. Thus, for each set of values for e, g^k obeying (A.2) there is one and only one set of four numbers β_λ obeying $\beta_0 > |\boldsymbol{\beta}|$ for which (A.1) is satisfied. Subsequently for any $n > 0$ there is a single real α for which also (4.8a) is satisfied.

ACKNOWLEDGMENTS

I am indebted to W. Israel, S. R. de Groot, and C. G. van Weert for discussions and to H. van Beijeren for a critical perusal of the manuscript. They should not be held responsible for the views expressed.

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